(3) 正态分布

若X的密度函数为

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty<x<+\infty$$

μ,σ 为常数 $,\sigma>0$

则称 X 服从参数为 μ,σ^2 的正态分布

记作 $X\sim N(\mu,\sigma^2)$

欧拉-泊松积分

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_{-\infty}^{+\infty} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\} dx \overset{\frac{x-\mu}{\sqrt{2}\sigma}=y}{=} \int_{-\infty}^{+\infty} \exp\{-y^2\}\sqrt{2}\sigma dy$$

$$= \sqrt{2}\sigma \cdot \sqrt{\pi} \quad = \sigma\sqrt{2\pi}$$

于是

$$\int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\} dx = 1$$

令 $I=\int_{-\infty}^{+\infty}e^{-x^2}dx \qquad I=\int_{-\infty}^{+\infty}e^{-y^2}dy$

$$I^2=\int_{-\infty}^{+\infty}e^{-x^2}dx \cdot \int_{-\infty}^{+\infty}e^{-y^2}dy$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-x^2}\cdot e^{-y^2}dxdy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-(x^2+y^2)}dxdy$$

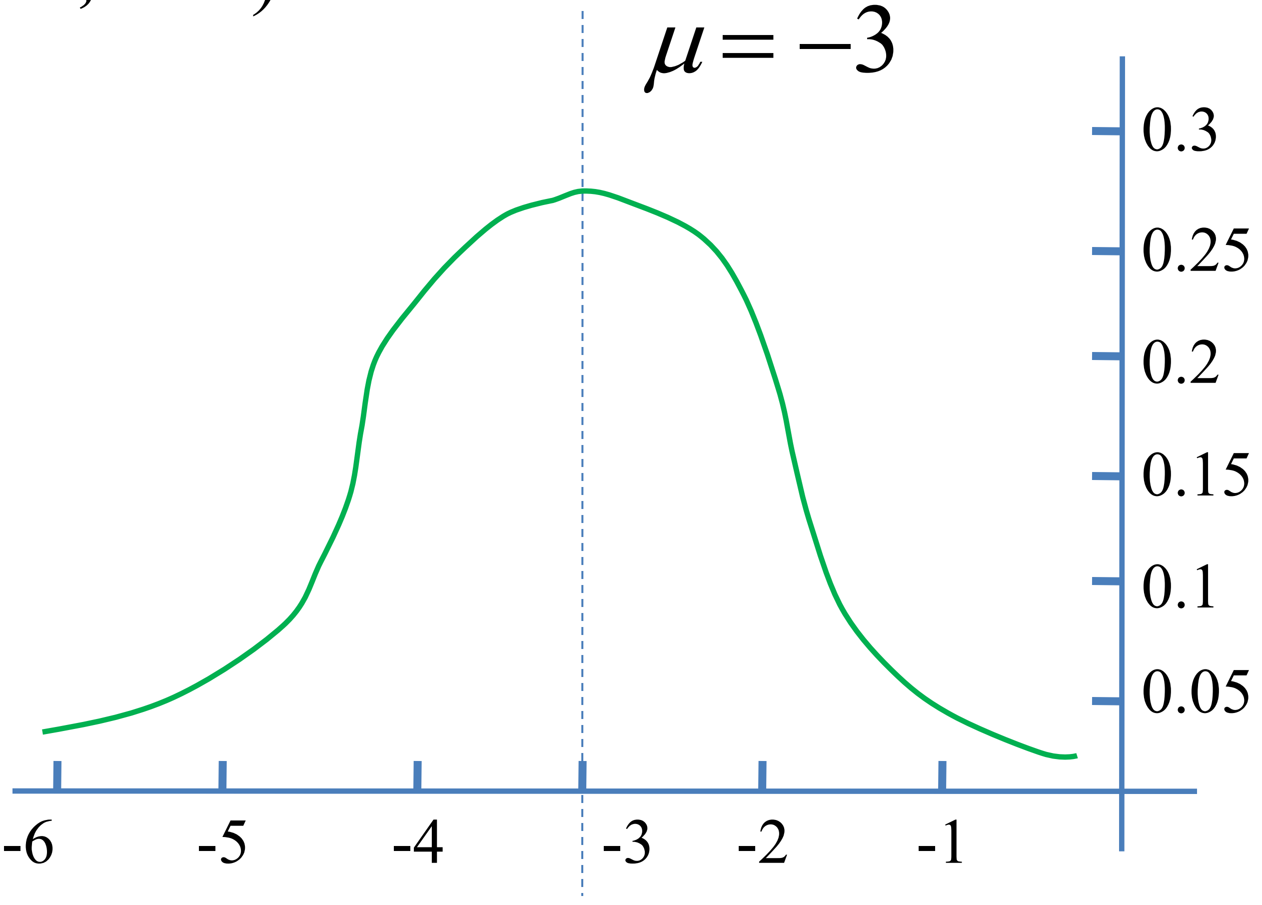
$$\overset{\substack{x=r\cos\theta \\ y=r\sin\theta}}{=} \int_0^{2\pi}\int_0^{+\infty}e^{-r^2}rdrd\theta = 2\pi\int_0^{+\infty}(-\frac{1}{2}e^{-r^2})'dr$$

$$=2\pi(-\frac{1}{2}e^{-r^2})\Big|_0^{+\infty}=2\pi\cdot\frac{1}{2}=\pi$$

于是 $I=\int_{-\infty}^{+\infty}e^{-x^2}dx=\sqrt{\pi}$

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < +\infty$$

$N(-3, 1.2)$



$f(x)$ 的性质：

1、图形关于直线 $x=\mu$ 对称：$f(\mu+x)=f(\mu-x)$

$$f(\mu+x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(\mu+x-\mu)^2}{2\sigma^2}} \quad =\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(x)^2}{2\sigma^2}}$$

$$f(\mu-x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(\mu-x-\mu)^2}{2\sigma^2}} \quad =\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(-x)^2}{2\sigma^2}}$$

$$f(\mu+x)=f(\mu-x)$$

图形关于直线 $x=\mu$ 对称

2、在 $x=\mu$ 时, $f(x)$ 取得最大值

$$\frac{1}{\sqrt{2\pi}\sigma}$$

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < +\infty$$

$$f'(x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\cdot\left(-\frac{2(x-\mu)}{2\sigma^2}\right)=0$$

3、在 $x=\mu\pm\sigma$ 时, 曲线 $y=f(x)$ 有拐点。

$$f'(x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\cdot\left(-\frac{2(x-\mu)}{2\sigma^2}\right)$$

$$=-\frac{1}{\sqrt{2\pi}\,\sigma\cdot\sigma^2}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\cdot(x-\mu)$$

$$f''(x)=-\frac{1}{\sqrt{2\pi}\,\sigma\cdot\sigma^2}\left[e^{-\frac{(x-\mu)^2}{2\sigma^2}}\left(-\frac{2(x-\mu)}{2\sigma^2}\right)\cdot(x-\mu)+e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right]$$

$$=-\frac{1}{\sqrt{2\pi}\,\sigma\cdot\sigma^2}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\left[-\frac{(x-\mu)^2}{\sigma^2}+1\right]=0$$

得拐点 $x=\mu\pm\sigma$

4、曲线 $y=f(x)$ 以x轴为渐近线

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < +\infty$$

$$\lim_{x\to\pm\infty} f(x)=0$$

5、曲线 $y=f(x)$ 的图形呈单峰状

$$F(\mu) = ?$$

$$\int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \overset{x=2\mu-y}{=} -\int_{+\infty}^{\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\mu-y)^2}{2\sigma^2}} dy$$

$$= \int_{\mu}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

$$= \int_{\mu}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

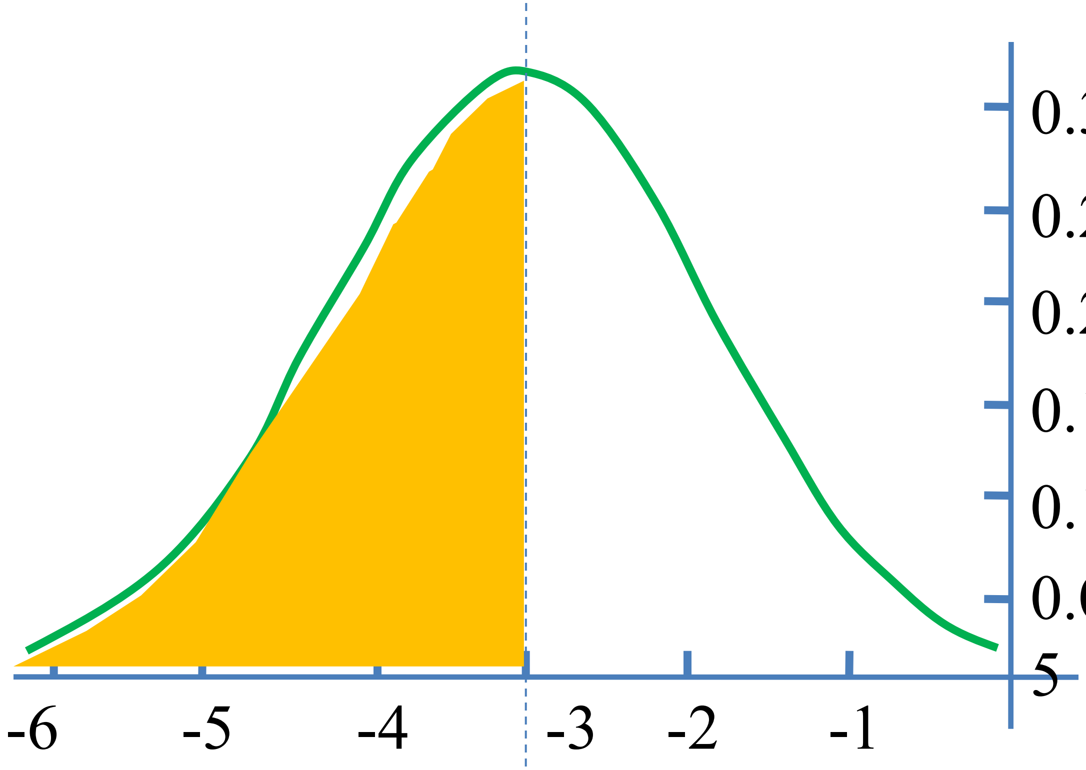
$$P(X \le \mu) = P(X > \mu) = \frac{1}{2}$$

$$F(\mu) = 1 - F(\mu) = \frac{1}{2}$$

$$P(X \le \mu) = F(\mu)$$

$$= 1 - F(\mu) = P(X > \mu)$$

$$= \frac{1}{2}$$



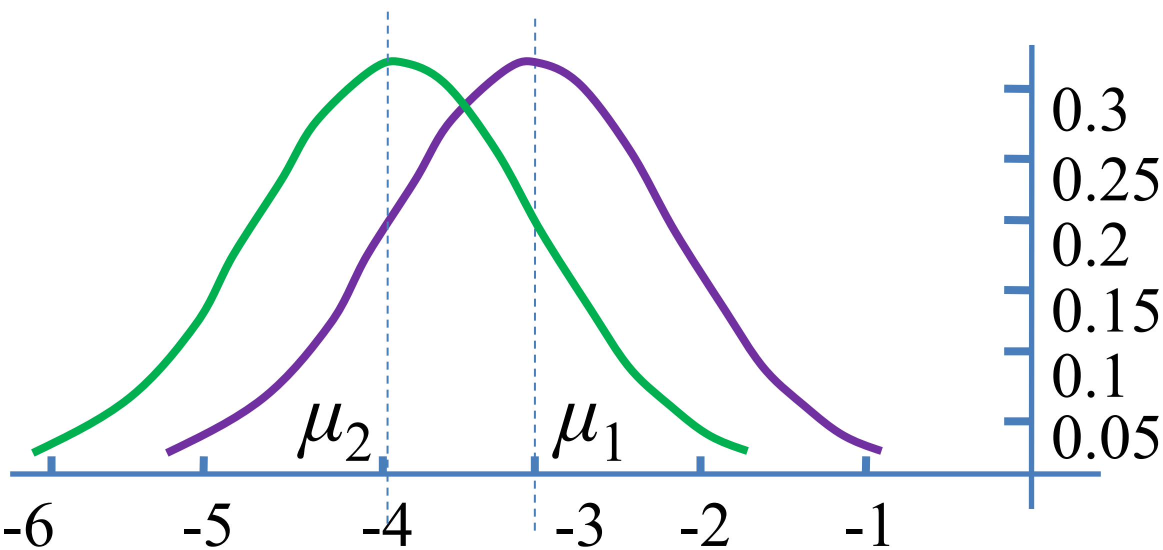
❑ $f(x)$ 的两个参数的含义：$f(x)=\dfrac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

μ — 位置参数

对于不同的 μ ，$f(x)$位置不同，但形状不变化。

最大值不变 $\dfrac{1}{\sqrt{2\pi}\,\sigma}$

对称轴与拐点之间的距离$|\mu - (\mu\pm\sigma)|$不变



$$f(x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

σ— 形状参数

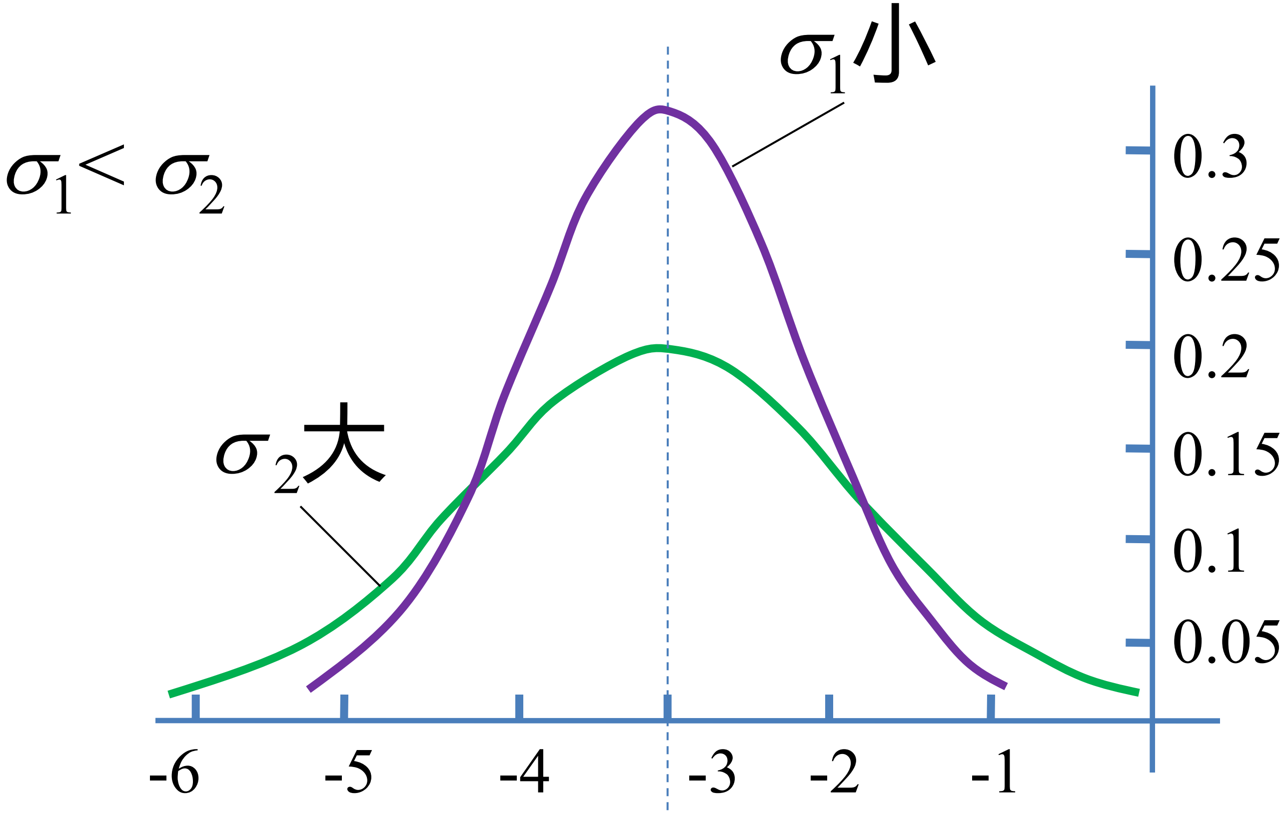
固定 μ ，对于不同的σ ，$f(x)$ 的形状不同.

若 $\sigma_1 < \sigma_2$ 则

$$f_{\sigma_1}(\mu)=\frac{1}{\sqrt{2\pi}\,\sigma_1} > \frac{1}{\sqrt{2\pi}\,\sigma_2}=f_{\sigma_2}(\mu)$$

前者取 μ附近值的概率更大.

$x=\mu\pm\sigma_1$ 所对应的拐点比$x=\mu\pm\sigma_2$ 所对应的拐点更靠近直线 $x=\mu$


$\sigma_1 < \sigma_2$
σ_1小
σ_2大
0.3
0.25
0.2
0.15
0.1
0.05
-6
-5
-4
-3
-2
-1

应用场合

若随机变量 X 受到众多相互独立的随机因素的影响，而每一个别因素的影响都是微小的，且这些影响可以叠加，则 X 服从正态分布.

可用正态变量描述的实例非常之多：

各种测量的误差；　人的生理特征；

工厂产品的尺寸；　农作物的收获量；

海洋波浪的高度；　金属线的抗拉强度；

热噪声电流强度；　学生们的考试成绩；

……　……

一种重要的正态分布：$N(0,1)$ — 标准正态分布

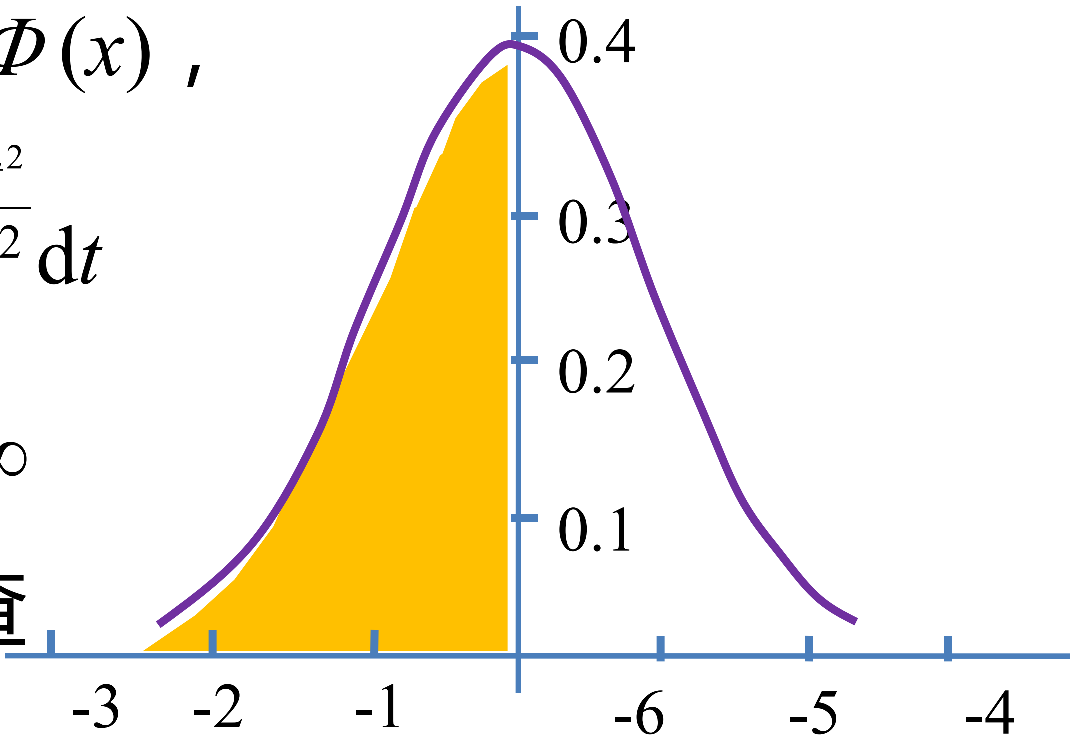
$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad -\infty < x < +\infty$$

$\varphi(x)$ 是偶函数，其图形关于纵轴对称

它的分布函数记为 $\Phi(x)$，

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t$$

$$-\infty < x < +\infty$$



其值有专门的表可查

$$\Phi(0) = 0.5$$

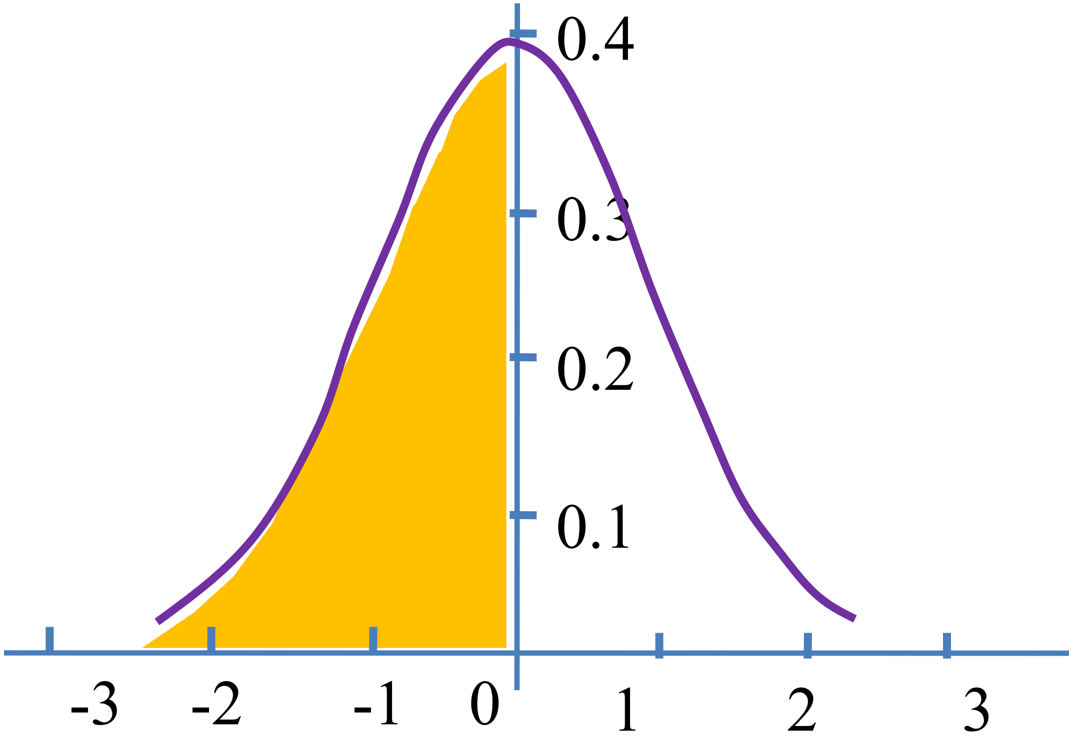
$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad -\infty < x < +\infty$$

$$\Phi(0) = \int_{-\infty}^{0} \varphi(t)dt$$

$$= \int_{0}^{+\infty} \varphi(t)dt$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \varphi(t)dt$$

$$= \frac{1}{2}$$



$$\Phi(-x) = 1 - \Phi(x)$$

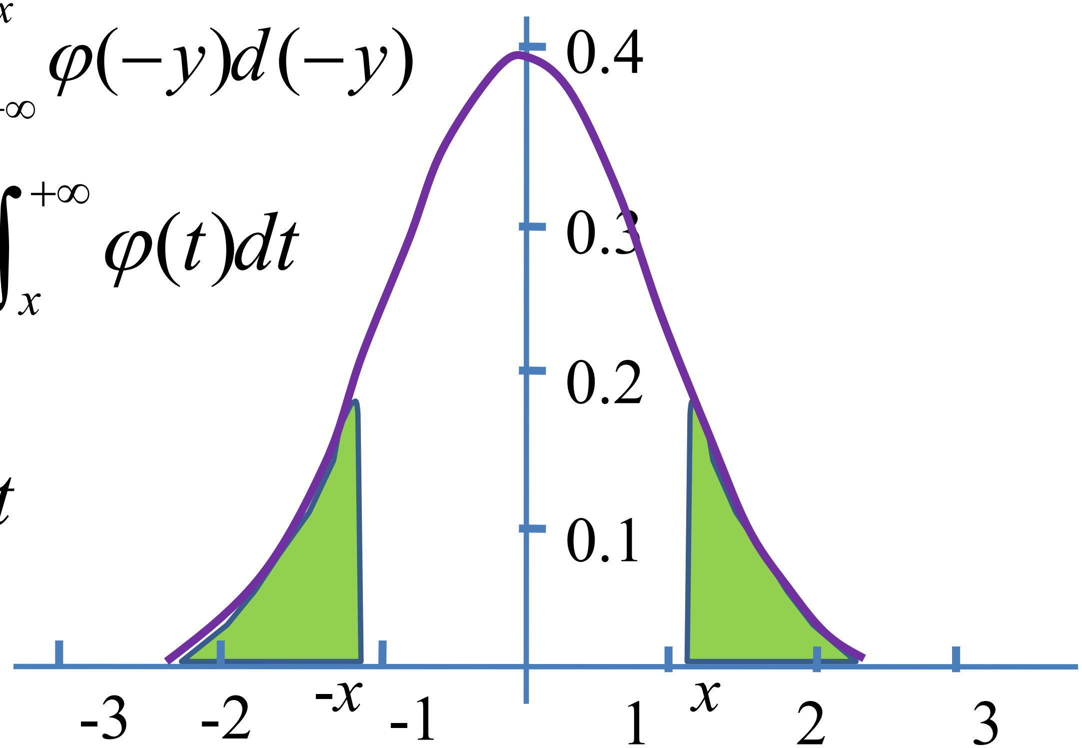
$t = - y$

$$\Phi(-x) = \int_{-\infty}^{-x} \phi(t)dt = \int_{+\infty}^{x} \varphi(-y)d(-y)$$

$$= -\int_{+\infty}^{x} \varphi(y)dy = \int_{x}^{+\infty} \varphi(t)dt$$

$$= 1 - \int_{-\infty}^{x} \varphi(t)dt$$

$$= 1 - \Phi(x)$$



$$P(|X| < a) = 2\Phi(a) - 1$$

对一般的正态分布 ：$X \sim N(\mu,\sigma^2)$

其分布函数 $F(x)=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\mathrm{d}t$

$$\overset{\frac{t-\mu}{\sigma}=y}{=} \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma}} \exp\{-\frac{y^2}{2}\}\sigma dy$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma}} \exp\{-\frac{y^2}{2}\}dy$$

$$=\int_{-\infty}^{\frac{x-\mu}{\sigma}} \varphi(y)dy$$

$$=\Phi(\frac{x-\mu}{\sigma})$$

$$P(a < X < b) = F(b) - F(a)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(X > a) = 1 - F(a)$$

$$= 1 - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

例1 设 $X \sim N(1,4)$，求 $P(0 \le X \le 1.6)$

解

$$P(0 \le X \le 1.6) = \Phi\left(\frac{1.6-1}{2}\right) - \Phi\left(\frac{0-1}{2}\right)$$

$$= \Phi(0.3) - \Phi(-0.5)$$

$$= \Phi(0.3) - [1 - \Phi(0.5)]$$

$$= 0.6179 - [1 - 0.6915]$$

$$= 0.3094$$

例2 已知 $X \sim N(2, \sigma^2)$ 且 $P(\,2 < X < 4\,) = 0.3,$

求 $P\,(\,X < 0\,).$

解一 $P(X < 0) = \Phi\left(\dfrac{0-2}{\sigma}\right) = 1 - \Phi\left(\dfrac{2}{\sigma}\right)$

$$P(2 < X < 4) = \Phi\left(\frac{4-2}{\sigma}\right) - \Phi\left(\frac{2-2}{\sigma}\right)$$

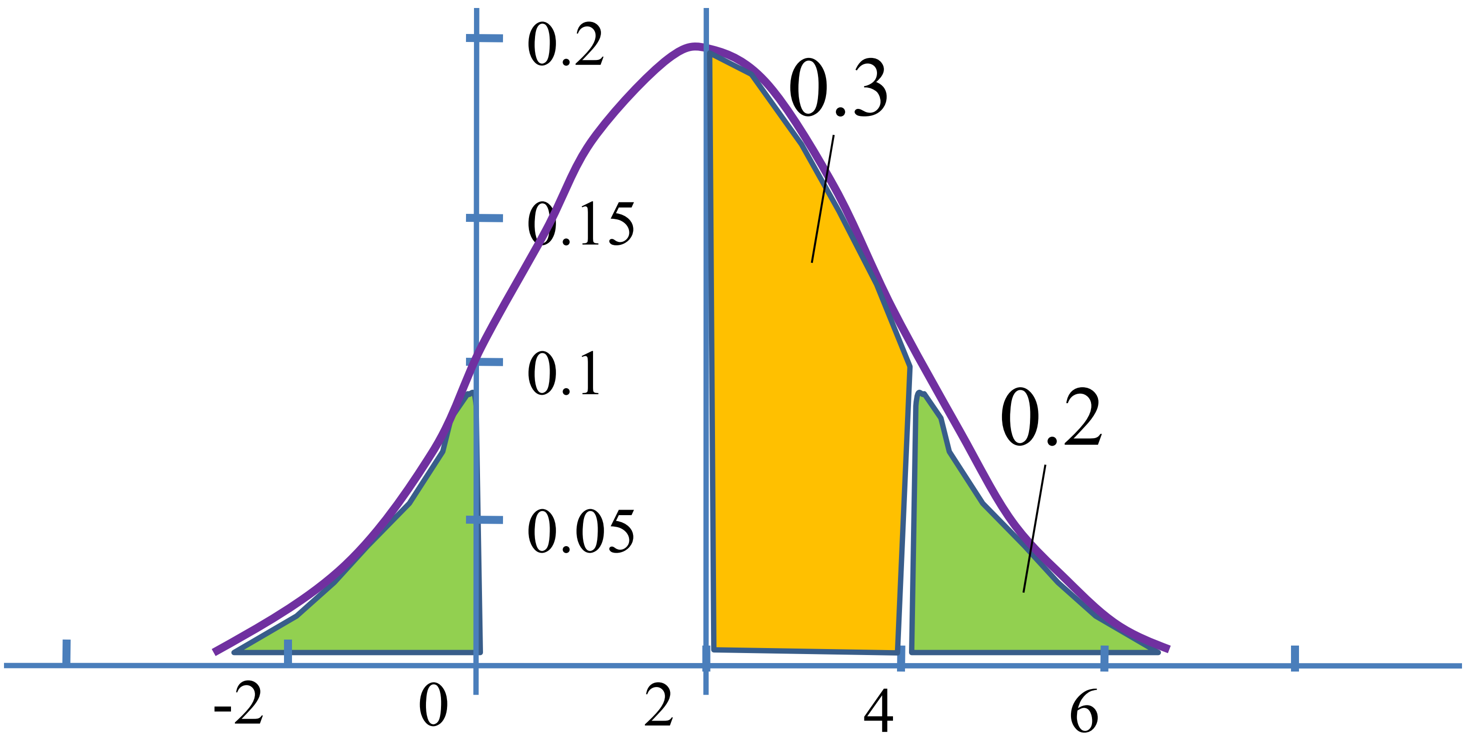
$$= \Phi\left(\frac{2}{\sigma}\right) - \Phi(0) = 0.3$$

$\Longrightarrow$ $\Phi\left(\dfrac{2}{\sigma}\right) = 0.8$

$\Longrightarrow$ $P(X < 0) = 0.2$

解二 图解法 $X \sim N(2,\sigma^2)$

$$P(\ 2 < X < 4\ ) = 0.3,$$



由图

$$P(X < 0) = 0.2$$

例3. 3σ 原理

设 $X \sim N(\mu, \sigma^2)$, 求 $P(|X-\mu|<3\sigma)$

解 $P(|X-\mu|<3\sigma) = P(\mu-3\sigma < X < \mu+3\sigma)$

$$= \Phi\left(\frac{\mu+3\sigma-\mu}{\sigma}\right) - \Phi\left(\frac{\mu-3\sigma-\mu}{\sigma}\right)$$

$$= \Phi(3) - \Phi(-3)$$

$$= 2\Phi(3) - 1 \quad = 2\times 0.9987 - 1 = 0.9974$$

在一次试验中, X 落入区间$(\mu-3\sigma, \mu+3\sigma)$的概率为 0.9974, 而超出此区间的可能性很小

由3σ 原理知,

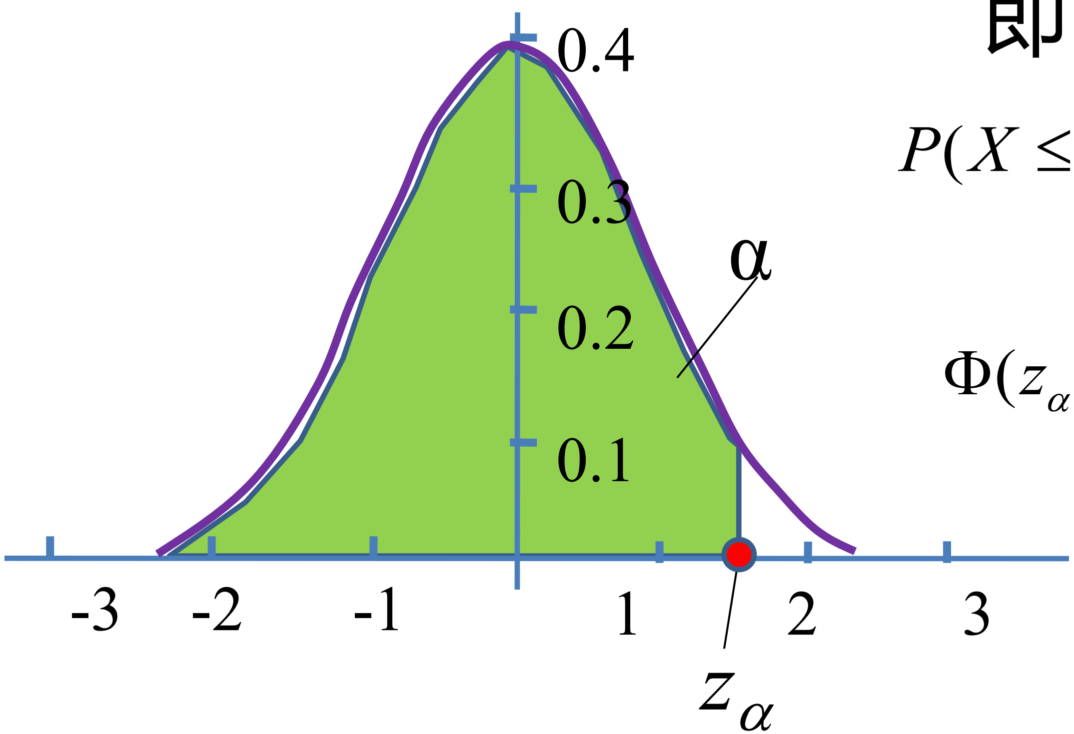
当 $a < -3$ 时 $\Phi(a) \approx 0$, $b > 3$ 时 $\Phi(b) \approx 1$

标准正态分布的 α 分位点 z_α

设 $X \sim N(0,1)$，$0 < \alpha < 1$，称满足

$$P(X \le z_\alpha) = \alpha$$

的点 z_α 为 X 的 α 分位点



即：

$$P(X \le z_\alpha) = \Phi(z_\alpha) = \alpha$$

$$\Phi(z_\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_\alpha} e^{-\frac{t^2}{2}} \mathrm{d}t$$

显然 $z_0 = -\infty,$

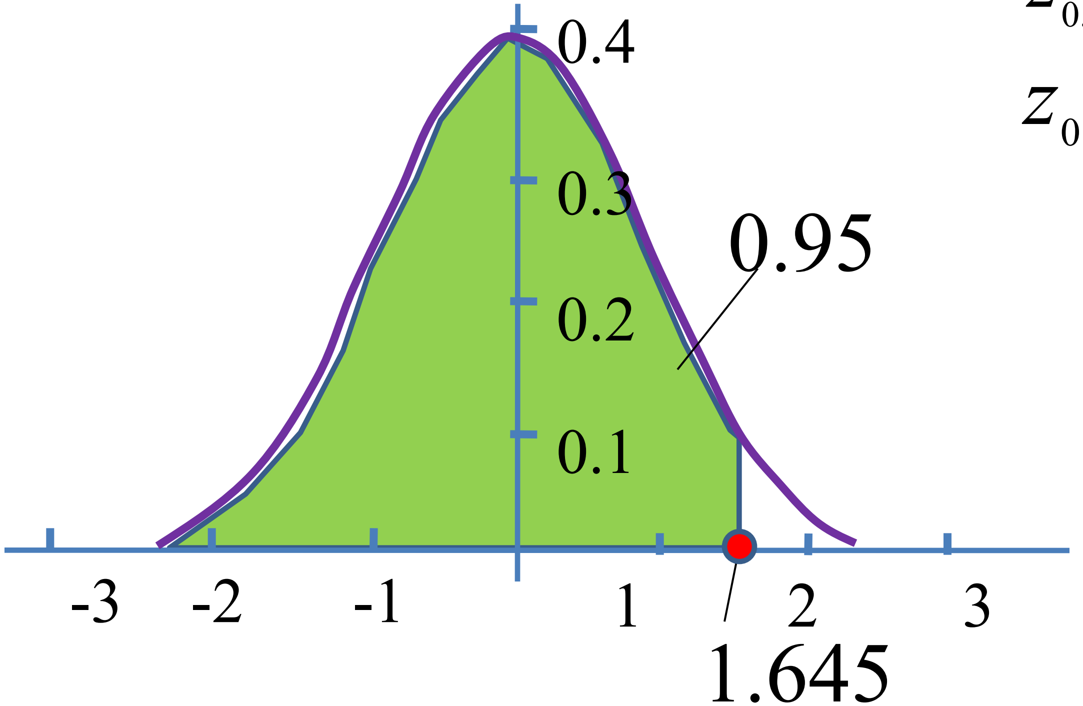
$z_{0.5} = 0,$

$z_1 = +\infty$

常用的几个数据

$z_{0.95} = 1.645$

$z_{0.975} = 1.96$



分位点的性质: $(0 < \alpha < 1)$

$$\Phi(z_\alpha) = P\{X \le z_\alpha\} = \alpha$$

$$\Phi(z_{1-\alpha}) = P\{X \le z_{1-\alpha}\} = 1-\alpha$$

$$\Phi(z_{1-\frac{\alpha}{2}}) = P\{X \le z_{1-\frac{\alpha}{2}}\} = 1-\frac{\alpha}{2}$$

(1) $z_\alpha = -z_{1-\alpha}$

事实上,

$$\Phi(z_\alpha) + \Phi(-z_\alpha) = 1$$

$$\Phi(z_\alpha) + \Phi(z_{1-\alpha}) = \alpha + (1-\alpha) = 1$$

得 $\Phi(-z_\alpha) = \Phi(z_{1-\alpha})$

于是 $-z_\alpha = z_{1-\alpha}$

$$z_\alpha = -z_{1-\alpha}$$

(2) $P\{X > z_{1-\alpha}\} = \alpha$ 其中 $z_{1-\alpha} > 0$

事实上,

$$P\{X > z_{1-\alpha}\} = 1 - P\{X \le z_{1-\alpha}\}$$

$$= 1 - \Phi(z_{1-\alpha})$$

$$= 1 - (1 - \alpha)$$

$$= \alpha$$

(3) $P\{|X|>z_{1-\frac{\alpha}{2}}\}=\alpha$

或 $P\{|X|\le z_{1-\frac{\alpha}{2}}\}=1-\alpha$

事实上,

$$P\{|X|\le z_{1-\frac{\alpha}{2}}\}=P\{-z_{1-\frac{\alpha}{2}}\le X\le z_{1-\frac{\alpha}{2}}\}$$

$$=\Phi(z_{1-\frac{\alpha}{2}})-\Phi(-z_{1-\frac{\alpha}{2}})$$

$$=\Phi(z_{1-\frac{\alpha}{2}})-\Phi(z_{\frac{\alpha}{2}})$$

$$=1-\frac{\alpha}{2}-\frac{\alpha}{2}=1-\alpha$$

$$P\{|X|>z_{1-\frac{\alpha}{2}}\}$$

$$=1-P\{|X|\le z_{1-\frac{\alpha}{2}}\}$$

$$=1-(1-\alpha)=\alpha$$

例4 设随机变量 $X \sim N(2,4^2)$

(1)求$P\{-3 \le X \le 5\}$

(2)求a,使$P\{|X-a|>a\}=0.7583$

解 (1) $P\{-3 \le X \le 5\}$

$$= F(5) - F(-3)$$

$$= \Phi(\frac{5-2}{4}) - \Phi(\frac{-3-2}{4})$$

$$= \Phi(0.75) - \Phi(-1.25)$$

$$= 0.7734 - 0.1056 = 0.6678$$

(2) $P\{|X-a|>a\}$

$=1-P\{|X-a|\leq a\}$

$=1-P\{-a\leq X-a\leq a\}$

$=1-P\{0\leq X\leq 2a\}$

$=1-[F(2a)-F(0)]$

$=1-[\Phi(\frac{2a-2}{4})-\Phi(\frac{0-2}{4})]$

$=1-\Phi(\frac{a}{2}-0.5)+\Phi(-0.5)$

$=1-\Phi(\frac{a}{2}-0.5)+0.3085$

$=0.7583$

$X\sim N(2,4^2)$

$$=1.3085-\Phi(\frac{a}{2}-0.5)$$

$$=0.7583$$

得 $\Phi(\frac{a}{2}-0.5)=0.5502$

$$\frac{a}{2}-0.5=0.125$$

$$a=1.25$$

例5 设随机变量 $X \sim N(\mu, \sigma^2)$

试用分位点表示下列常数a, b

(1) $\mu = 0, \sigma = 1,\quad P\{-X < a\} = 0.025$

(2) $\mu = 1, \sigma = 2,\quad P\{|X - 1| \le b\} = 0.75$

解(1) $X \sim N(0,1)$

$$P\{-X < a\} = P\{X > -a\}$$

$$= 1 - P\{X \le -a\} = 0.025$$

$$P\{X \le -a\} = 1 - 0.025 = 0.975 = P\{X \le z_{0.975}\}$$

因此, $-a = z_{0.975}$ $\quad a = -z_{0.975} = z_{0.025}$

(2) $X \sim N(1, 2^2)$ $\qquad P\{|X-1| \le b\} = 0.75$

$$P\{|X-1| \le b\} = P\{1-b \le X \le 1+b\}$$

$$= F(1+b) - F(1-b)$$

$$= \Phi(\frac{1+b-1}{2}) - \Phi(\frac{1-b-1}{2})$$

$$= \Phi(\frac{b}{2}) - \Phi(-\frac{b}{2})$$

$$= \Phi(\frac{b}{2}) - [1 - \Phi(\frac{b}{2})]$$

$$= 2\Phi(\frac{b}{2}) - 1 = 0.75$$

$$2\Phi(\frac{b}{2}) = 1.75$$

$$\Phi(\frac{b}{2}) = 0.875 \ = \Phi(z_{0.875})$$

$$\frac{b}{2} = z_{0.875}$$

故 $b = 2z_{0.875}$

例6 已知随机变量 $X \sim N(2,\sigma^2)$

且 $P\{|X-3|\le 1\}=0.44$

求 $P\{|X-2|\ge 2\}$

解 $P\{|X-3|\le 1\}=P\{-1\le X-3\le 1\}$

$$=P\{2\le X\le 4\}$$

$$=F(4)-F(2)$$

$$=\Phi(\frac{4-2}{\sigma})-\Phi(\frac{2-2}{\sigma})$$

$$=\Phi(\frac{2}{\sigma})-\Phi(0)$$

$$=\Phi(\frac{2}{\sigma})-0.5=0.44$$

从而， $\Phi(\frac{2}{\sigma}) = 0.94$

$$P\{|X-2|\geq 2\}$$

$$=1-P\{|X-2|<2\}$$

$$=1-P\{0<X<4\}$$

$$=1-[F(4)-F(0)]$$

$$=1-[\Phi(\frac{4-2}{\sigma})-\Phi(\frac{-2}{\sigma})]$$

$$=1-[\Phi(\frac{2}{\sigma})-\Phi(-\frac{2}{\sigma})]$$

$$=1-\Phi(\frac{2}{\sigma})+[1-\Phi(\frac{2}{\sigma})]$$

$$=2[1-\Phi(\frac{2}{\sigma})]=2(1-0.94)$$

$$=2\times 0.06=0.12$$

例7 设测量的误差 $X \sim N(7.5, 100)$(单位：米),问要进行多少次独立测量，才能使至少有一次误差的绝对值不超过10米的概率大于0.9？

解

$$P(|X| \le 10) = \Phi\left(\frac{10-7.5}{10}\right) - \Phi\left(\frac{-10-7.5}{10}\right)$$
$$= \Phi(0.25) - \Phi(-1.75)$$
$$= \Phi(0.25) - [1 - \Phi(1.75)]$$
$$= 0.5586$$

设 A 表示进行 n 次独立测量至少有一次误差的绝对值不超过10米

$$P(A) = 1 - (1 - 0.5586)^n > 0.9 \quad \Rightarrow \quad n > 3$$

所以至少要进行 4 次独立测量才能满足要求.